Post-Newtonian results in the analytical treatment of compact binaries

G.Schaefer@tpi.uni-jena.de

Outline

- ADM Hamiltonian and Routh Functional
- Brill-Lindquist vs Misner-Lindquist BHs
- Post-Newtonian Expansions
- 3PN Binary BH Conservative Dynamics
- Dynamical Invariants
- ISCO
- Inspiralling

ADM Hamiltonian and Routh Functional

$$g^{1/2}\mathrm{R} = \frac{1}{g^{1/2}}\left(\pi^i_j\pi^j_i - \frac{1}{2}\pi^i_i\pi^j_j\right) + \frac{16\pi G}{c^3}\sum_a \left(m_a^2c^2 + g^{ij}p_{ai}p_{aj}\right)^{1/2}\delta_a$$

$$-2\partial_j\pi^j_i + \pi^{kl}\partial_i g_{kl} = \frac{16\pi G}{c^3}\sum_a p_{ai}\delta_a$$

3 Cordinate Conditions: $g_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h^{\mathrm{TT}}_{ij}$

CC: $\pi^{ii} = 0, \qquad \pi^{ij} = -g^{1/2}(K^{ij} - g^{ij}K), \quad \pi^i_i = \pi^{ij}h^{\mathrm{TT}}_{ij}$

---

unique decomposition: $\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij}_{\mathrm{TT}}$

$$\tilde{\pi}^{ij} = \partial_i\pi^j + \partial_j\pi^i - \frac{2}{3}\delta_{ij}\partial_k\pi^k$$

$\pi^{ij}_{\mathrm{TT}}c^3/16\pi G$: canonical conjugate to h^{TT}_{ij}

ADM Hamiltonian

$$H\left[x^i_a, p_{ai}, h^{\mathrm{TT}}_{ij}, \pi^{ij}_{\mathrm{TT}}\right] = -\frac{c^4}{16\pi G}\int d^3x\ \Delta\phi\left[x^i_a, p_{ai}, h^{\mathrm{TT}}_{ij}, \pi^{ij}_{\mathrm{TT}}\right]$$

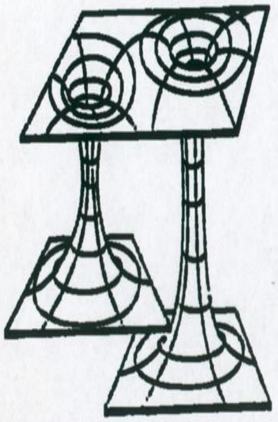
Routh functional

$$R\left[x^i_a, p_{ai}, h^{\mathrm{TT}}_{ij}, \partial_t h^{\mathrm{TT}}_{ij}\right] = H - \frac{c^3}{16\pi G}\int d^3x\ \pi^{ij}_{\mathrm{TT}}\partial_t h^{\mathrm{TT}}_{ij}$$

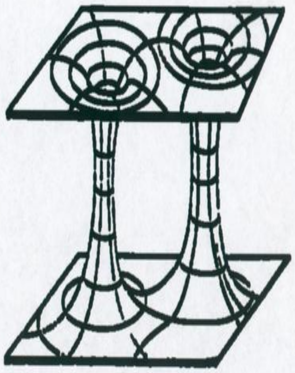
$$\frac{\delta\int R(t')dt'}{\delta h^{\mathrm{TT}}_{ij}(x^k, t)} = 0, \quad \dot{p}_{ai} = -\frac{\partial R}{\partial x^i_a}, \quad \dot{x}^i_a = \frac{\partial R}{\partial p_{ai}}$$

Konfigurationen binärer Schwarzer Löcher

Brill-Lindquist



Misner-Lindquist



Brill-Lindquist vs Misner-Lindquist BHs

$$-\left(1+\frac{1}{8}\,\phi\right)\Delta\phi = \frac{16\pi G}{c^2}\sum_a m_a\delta_a \qquad \left(h_{ij}^{\mathrm{TT}} = 0 = p_{ai}\right)$$

$$\phi = \frac{4G}{c^2}\left(\frac{\alpha_1}{r_1}+\frac{\alpha_2}{r_2}\right)$$ unique solution

$$\alpha_a = m_a - \frac{m_a+m_b}{2} + \frac{c^2 r_{ab}}{G}\left(\sqrt{1+\frac{m_a+m_b}{c^2 r_{ab}/G}+\left(\frac{m_a-m_b}{2c^2 r_{ab}/G}\right)^2}-1\right)$$

$$H_{\mathrm{BL}} = (\alpha_1+\alpha_2)\,c^2 = (m_1+m_2)\,c^2 - G\,\frac{\alpha_1\alpha_2}{r_{12}}$$

$$H^{3PN}_{\text{ambiguous}} = \omega_{\text{static}}(m_1+m_2)\frac{G^4 m_1^2 m_2^2}{c^6 r_{12}^4} = \omega_{\text{static}}\frac{G^4 M^3 \mu^2}{c^6 r_{12}^4}$$

$\omega_{\text{static}} = 0$ BL, $\omega_{\text{static}} = -\frac{1}{8}$ ML (shift of positions

$$H^{3PN}_{\text{induced}} = -\omega_{\text{static}}\left(\mathbf{p}^2 - 3(\mathbf{n}\cdot\mathbf{p})^2\right)\frac{G^3 M^2}{c^6 r_{12}^3}$$ relative to BL)

Thus, coordinate (gauge) transformation (given in CMS):

$$\delta H = \omega_{\text{static}}\left(\mathbf{p}^2 - 3(\mathbf{n}\cdot\mathbf{p})^2\right)\frac{G^3 M^2}{c^6 r_{12}^3} - \omega_{\text{static}}\frac{G^4 M^3 \mu^2}{c^6 r_{12}^4}$$

Post-Newtonian Expansions

$$R\left[x_a^i, p_{ai}, h_{ij}^{\text{TT}}, \partial_t h_{ij}^{\text{TT}}\right] - Mc^2 = \sum_{n=0}^{\infty}\left(\frac{1}{c^2}\right)^n R_n\left[x_a^i, p_{ai}, \hat{h}_{ij}^{\text{TT}}, \partial_t \hat{h}_{ij}^{\text{TT}}\right]$$

$$h_{ij}^{\text{TT}} = \tfrac{G}{c^4}\hat{h}_{ij}^{\text{TT}}$$

$$\left(\Delta - \frac{\partial_t^2}{c^2}\right)h^{\text{TT}} = \frac{G}{c^4}\sum_{n=0}^{\infty}\left(\frac{1}{c^2}\right)^n D_n^{\text{TT}}[x, x_a(t), p_a(t), \hat{h}^{\text{TT}}(t), \partial_t \hat{h}^{\text{TT}}(t)]$$

Radiation field:

$$\begin{aligned} h^{\mathrm{TT}}_{ij}(\mathbf{x},t) &= \frac{G}{c^4}\frac{P_{ijkm}(\mathbf{n})}{r}\sum_{l=2}^{\infty}\left\{\left(\frac{1}{c^2}\right)^{\frac{l-2}{2}}\frac{4}{l!}\,\mathrm{M}^{[l]}_{kmi_3\ldots i_l}(t-\frac{r_*}{c})\,N_{i_3\ldots i_l}\right. \\ &+ \left.\left(\frac{1}{c^2}\right)^{\frac{l-1}{2}}\frac{8l}{(l+1)!}\,\epsilon_{pq(k}\,\mathrm{S}^{[l]}_{m)pi_3\ldots i_l}(t-\frac{r_*}{c})\,n_q\,N_{i_3\ldots i_l}\right\} \end{aligned}$$

Multipole moment with tail:

$$\begin{aligned} \mathrm{M}_{ij}(t-\frac{r_*}{c}) &= \widehat{\mathrm{M}}_{ij}\left(t-\frac{r_*}{c}\right) \\ &+ \frac{2Gm}{c^3}\int_0^\infty dv\,\ln\left(\frac{v}{2b}\right)\widehat{\mathrm{M}}^{[2]}_{ij}(t-\frac{r_*}{c}-v)+O(1/c^4) \end{aligned}$$

$$r_* = r+\frac{2Gm}{c^2}\ln\left(\frac{r}{cb}\right)+O(1/c^3)$$

Luminosity:

$$\mathcal{L}=\frac{G}{5c^5}\sum_{n=0}^{\infty}\left(\frac{1}{c^2}\right)^n\hat{\mathcal{L}}_n$$

2PN energy loss:

$$-<\frac{d\mathcal{E}(t-r_*/c)}{dt}> \;=\; <\mathcal{L}(t)>$$

$$\begin{aligned} \mathcal{L} &= \frac{G}{5c^5}\left\{\mathrm{M}^{[3]}_{ij}\mathrm{M}^{[3]}_{ij}+\frac{1}{c^2}\left[\frac{5}{189}\mathrm{M}^{[4]}_{ijk}\mathrm{M}^{[4]}_{ijk}+\frac{16}{9}\mathrm{S}^{[3]}_{ij}\mathrm{S}^{[3]}_{ij}\right]\right. \\ &+ \left.\frac{1}{c^4}\left[\frac{5}{9072}\mathrm{M}^{[5]}_{ijkm}\mathrm{M}^{[5]}_{ijkm}+\frac{5}{84}\mathrm{S}^{[4]}_{ijk}\mathrm{S}^{[4]}_{ijk}\right]\right\} \end{aligned}$$

3PN Binary BH Conservative Dynamics

$$\begin{aligned} H(t) &= m_1c^2 + m_2c^2 + H_N + \frac{1}{c^2}H_{[1PN]} \\ &+ \frac{1}{c^4}H_{[2PN]} + \frac{1}{c^6}H_{[3PN]} + ... \\ &+ \frac{1}{c^5}H_{[2.5PN]}(t) + \frac{1}{c^7}H_{[3.5PN]}(t) + ... \end{aligned}$$

$\hat{H} = (H - Mc^2)/\mu, \quad \mu = m_1m_2/M, \quad M = m_1 + m_2$

$\nu = \mu/M, \quad 0 \leq \nu \leq 1/4$

test-body case: $\nu = 0$, equal-mass case: $\nu = 1/4$

CMS: $\mathbf{p}_1 + \mathbf{p}_2 = 0, \quad \mathbf{p} \equiv \mathbf{p}_1/\mu,$

$p_r = (\mathbf{n}\cdot\mathbf{p}), \quad \mathbf{q} \equiv (\mathbf{x}_1 - \mathbf{x}_2)/GM, \quad \mathbf{n} = \mathbf{q}/|\mathbf{q}|$

$$\hat{H}_N = \frac{p^2}{2} - \frac{1}{q}$$

$$\hat{H}_{[1PN]} = \frac{1}{8}(3\nu - 1)p^4 - \frac{1}{2}[(3+\nu)p^2 + \nu p_r^2]\frac{1}{q} + \frac{1}{2q^2}$$

$$\begin{aligned} \hat{H}_{[2PN]} &= \frac{1}{16}(1 - 5\nu + 5\nu^2)p^6 \\ &+ \frac{1}{8}[(5 - 20\nu - 3\nu^2)p^4 - 2\nu^2p_r^2p^2 - 3\nu^2p_r^4]\frac{1}{q} \\ &+ \frac{1}{2}[(5+8\nu)p^2 + 3\nu p_r^2]\frac{1}{q^2} - \frac{1}{4}(1+3\nu)\frac{1}{q^3} \end{aligned}$$

$$
\begin{aligned}
\hat{H}_{[3PN]} &= \frac{1}{128}(-5+35\nu-70\nu^2+35\nu^3)p^8 \\
&+ \frac{1}{16}[(-7+42\nu-53\nu^2-5\nu^3)p^6+(2-3\nu)\nu^2 p_r^2 p^4 \\
&+ 3(1-\nu)\nu^2 p_r^4 p^2 - 5\nu^3 p_r^6]\frac{1}{q} \\
&+ [\frac{1}{16}(-27+136\nu+109\nu^2)p^4+\frac{1}{16}(17+30\nu)\nu p_r^2 p^2 \\
&+ \frac{1}{12}(5+43\nu)\nu p_r^4]\frac{1}{q^2} \\
&+ [\left(-\frac{25}{8}+\left(\frac{1}{64}\pi^2-\frac{335}{48}\right)\nu-\frac{23}{8}\nu^2\right)p^2 \\
&+ \left(-\frac{85}{16}-\frac{3}{64}\pi^2-\frac{7}{4}\nu\right)\nu p_r^2]\frac{1}{q^3}+[\frac{1}{8}+\left(\frac{109}{12}-\frac{21}{32}\pi^2\right)\nu]\frac{1}{q^4}
\end{aligned}
$$

Third post-Newtonian accurate generalized quasi-Keplerian parametrization for compact binaries in eccentric orbits

Raoul-Martin Memmesheimer, Achamveedu Gopakumar
& Gerhard Schäfer

Physical Review D, 2004 (in press), **gr-qc/0407049**

- The equations of motion derivable from the 3PN accurate conservative Hamiltonian, in ADM–type coordinates in the center–of–mass frame, allows 'Keplerian type' parametric solution:

- The associated 3PN accurate orbital energy & angular momentum **L** are conserved.

- The conservation of **L** => that the motion is restricted to a plane, namely the orbital plane.

Introduce polar coordinates such that

$$\mathbf{r} = r(\cos\varphi, \sin\varphi).$$

- The radial motion is parametrized by

$$r = a_r (1 - e_r \cos u)$$

⋆ a_r and e_r are 3PN accurate semi-major axis & 'radial eccentricity'.

These are expressible in terms of orbital energy E, angular momentum L and m_1 and m_2.

⋆ u is the eccentric anomaly.

- The angular motion is given by

$$\begin{aligned}\varphi - \varphi_0 &= (1+k)\,v + \left(\frac{f_{4\varphi}}{c^4} + \frac{f_{6\varphi}}{c^6}\right)\sin 2v + \left(\frac{g_{4\varphi}}{c^4} + \frac{g_{6\varphi}}{c^6}\right)\sin 3v \\ &\quad + \frac{i_{6\varphi}}{c^6}\sin 4v + \frac{h_{6\varphi}}{c^6}\sin 5v\,,\end{aligned}$$

where $v = 2\arctan\left[\left(\frac{1+e_\varphi}{1-e_\varphi}\right)^{1/2}\tan\frac{u}{2}\right]$

⋆ v is the true anomaly

⋆ k is the measure of the advance of the periastron & e_φ is the 'angular eccentricity'

⋆ $f_{4\varphi}$, $f_{6\varphi}$, $g_{4\varphi}$, $g_{6\varphi}$, $i_{6\varphi}$, and $h_{6\varphi}$ are 2PN & 3PN order orbital functions expressible in terms of E, L, m_1 and m_2

- The 3PN accurate **'Kepler equation'**, which connects the eccentric anomaly to the coordinate time reads

$$\begin{aligned}l \equiv n\,(t - t_0) &= u - e_t \sin u + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6}\right)(v - u) \\ &\quad + \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6}\right)\sin v + \frac{i_{6t}}{c^6}\sin 2v + \frac{h_{6t}}{c^6}\sin 3v\end{aligned}$$

⋆ l is the mean anomaly, n the mean motion & e_t the 'time eccentricity'

⋆ g_{4t}, g_{6t}, f_{4t}, f_{6t}, i_{6t} & h_{6t} are 2PN & 3PN order orbital functions expressible in terms of E, L, m_1 and m_2

- Our work extends similar parametrizations obtained at 1PN & 2PN orders by Damour & Deruelle (1985), Damour & Schäfer (1988) and Schäfer & Wex (1993).
- The expressions for n & k, expressed in terms of E, L , m_1 and m_2 are **gauge invariant quantities.**
- The three eccentricities e_r, e_φ and e_t are connected by PN accurate relations involving E, L, m_1 and m_2.
- Our parametrization will be required to
  - **Construct search templates for binaries in eccentric orbits**
  - **Construct gauge invariant 'diagnostic tool' for general relativistic numerical simulations**
  - **Construct 'timing formula' relevant for Square Kilometer Array (SKA)**

Dynamical Invariants

radial action $i_r(E,j)$:

$$i_r(E,j) = \frac{1}{2\pi}\oint dr\, p_r$$

phase of revolution Φ:

$$\frac{\Phi}{2\pi} = 1 + k = -\frac{\partial}{\partial j} i_r(E,j)$$

orbital period P:

$$\frac{P}{2\pi GM} = \frac{\partial}{\partial E} i_r(E,j)$$

periastron advance k:

$$\begin{aligned} k &= \frac{1}{c^2}\frac{3}{j^2}\left\{1+\frac{1}{c^2}\left[\frac{5}{4}(7-2\nu)\frac{1}{j^2}+\frac{1}{2}(5-2\nu)\,E\right]\right. \\ &+ \left.\frac{1}{c^4}\left[a_1(\nu)\frac{1}{j^4}+a_2(\nu)\frac{E}{j^2}+a_3(\nu)\,E^2\right]\right\} \end{aligned}$$

$$\begin{aligned} \frac{P}{2\pi GM} &= \frac{1}{(-2E)^{3/2}}\left\{1-\frac{1}{c^2}\frac{1}{4}(15-\nu)E\right. \\ &+ \frac{1}{c^4}\left[\frac{3}{2}(5-2\nu)\frac{(-2E)^{3/2}}{j}-\frac{3}{32}(35+30\nu+3\nu^2)\,E^2\right] \\ &+ \left.\frac{1}{c^6}\left[a_2(\nu)\frac{(-2E)^{3/2}}{j^3}-3a_3(\nu)\frac{(-2E)^{5/2}}{j}+a_4(\nu)\,E^3\right]\right\} \end{aligned}$$

$$\begin{aligned} a_1(\nu) &= \frac{5}{2}\left(\frac{77}{2}+\left(\frac{41}{64}\pi^2-\frac{125}{3}\right)\nu+\frac{7}{4}\nu^2\right) \\ a_2(\nu) &= \frac{105}{2}+\left(\frac{41}{64}\pi^2-\frac{218}{3}\right)\nu+\frac{45}{6}\nu^2 \\ a_3(\nu) &= \frac{1}{4}(5-5\nu+4\nu^2) \\ a_4(\nu) &= \frac{5}{128}(21-105\nu+15\nu^2+5\nu^3) \end{aligned}$$

Effects of spinning objects

$$\mathbf{P}_1\delta(\mathbf{x}-\mathbf{x}_1) \to (\mathbf{P}_1 + \tfrac{1}{2}\mathbf{S}_1 \times \nabla_1)\delta(\mathbf{x}-\mathbf{x}_1)$$

$$\boxed{H_{\mathrm{SO}} = \frac{2G}{c^2R^3}(\mathbf{S}\cdot\mathbf{L}) + \frac{3GM_1M_2}{2c^2R^3}(\hat{\mathbf{S}}\cdot\mathbf{L})}$$

$$\mathbf{S} \equiv \mathbf{S}_1 + \mathbf{S}_2,\ \ \hat{\mathbf{S}} \equiv \frac{\mathbf{S}_1}{M_1^2} + \frac{\mathbf{S}_2}{M_2^2},\ \ \mathbf{L} \equiv \mathbf{R}\times\mathbf{P},\ \ \mathbf{P} \equiv \mathbf{P}_1 = -\mathbf{P}_2$$

$$\boxed{H_{\mathrm{SS}} = \frac{G}{c^2R^3}\Big(\frac{3(\mathbf{S}_1\cdot\mathbf{R})(\mathbf{S}_2\cdot\mathbf{R})}{R^2} - (\mathbf{S}_1\cdot\mathbf{S}_2)\Big)}$$

$$H_{\mathrm{SS}}^{\mathrm{Kerr}} = \frac{GM_1M_2}{2c^2R^3}\Big(\frac{3(\tilde{\mathbf{S}}\cdot\mathbf{R})(\tilde{\mathbf{S}}\cdot\mathbf{R})}{R^2} - (\tilde{\mathbf{S}}\cdot\tilde{\mathbf{S}})\Big), \qquad \tilde{\mathbf{S}} \equiv \frac{\mathbf{S}_1}{M_1} + \frac{\mathbf{S}_2}{M_2}$$

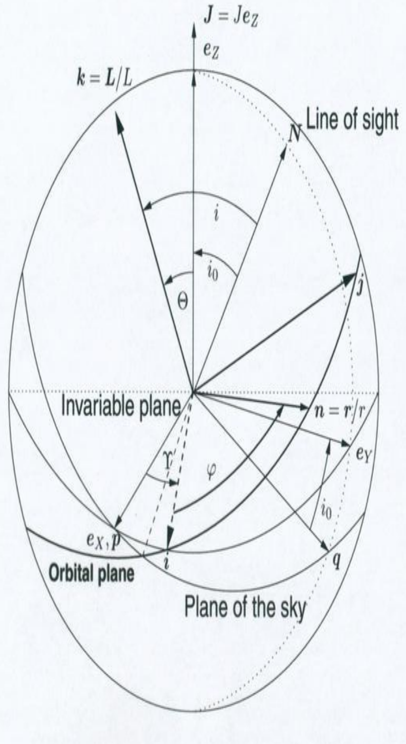
Post–Newtonian Results

Gerhard Schäfer
with C. Königsdörffer and A. Gopakumar
G.Schaefer@tpi.uni-jena.de

Theoretisch-Physikalisches Institut
Friedrich-Schiller-Universität
Jena, Germany
http://www.tpi.uni-jena.de

POST–NEWTONIAN ACCURATE PARAMETRIC SOLUTION TO THE DYNAMICS OF SPINNING COMPACT BINARIES IN ECCENTRIC ORBITS: I. THE LEADING ORDER SPIN–ORBIT INTERACTION

Binary geometry and orbital plane precession



Dynamics in (r, φ, Υ) in $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$

- solution for the relative motion: $\boldsymbol{r} = r\cos\varphi\,\boldsymbol{i} + r\sin\varphi\,\boldsymbol{j}$
- the basic vectors: $\begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix} = \mathcal{R}(\boldsymbol{i}, \Theta)\,\mathcal{R}(\boldsymbol{e}_Z, \Upsilon) \begin{pmatrix} \boldsymbol{e}_X \\ \boldsymbol{e}_Y \\ \boldsymbol{e}_Z \end{pmatrix}$
- the quasi–Keplerian representation:

$$r = a_r(1 - e_r\cos u)\,, \qquad n(t - t_0) = u - e_t \sin u\,,$$

$$\varphi - \varphi_0 = (1 + k)v\,,$$

$$\Upsilon - \Upsilon_0 = \frac{\chi_{\mathrm{so}} J}{c^2 L^3}(v + e\sin v)\,,$$

$$v = 2\arctan\left[\left(\frac{1 + e_\varphi}{1 - e_\varphi}\right)^{1/2} \tan\frac{u}{2}\right]$$

Dynamics in (r, φ, Υ) in $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$

- the associated parameters in terms of E, L, S, η and $\alpha = \angle(\boldsymbol{L}, \boldsymbol{S})$:

$$a_r = -\frac{1}{2E}\left(1 - 2\chi_{\mathrm{so}}\cos\alpha\frac{S}{L}\frac{E}{c^2}\right),$$

$$e_r^2 = 1 + 2EL^2 + 8(1 + EL^2)\chi_{\mathrm{so}}\cos\alpha\frac{S}{L}\frac{E}{c^2}\,,$$

$$n = (-2E)^{3/2}\,,$$

$$e_t^2 = 1 + 2EL^2 + 4\chi_{\mathrm{so}}\cos\alpha\frac{S}{L}\frac{E}{c^2}\,,$$

$$k = \frac{1}{c^2L^2}\left(\chi_{\mathrm{so}} - 3\chi_{\mathrm{so}}\cos\alpha\frac{S}{L}\right),$$

$$e_\varphi^2 = 1 + 2EL^2 - 4(1 + 2EL^2)\chi_{\mathrm{so}}\frac{E}{c^2}$$

$$+ 4(3 + 4EL^2)\chi_{\mathrm{so}}\cos\alpha\frac{S}{L}\frac{E}{c^2}\,.$$

The ISCO for Single Black Holes

$$\text{SBH:}\quad E(x) = \frac{1-2x}{(1-3x)^{1/2}} - 1$$

$$= -\frac{1}{2}x + \frac{3}{8}x^2 + \frac{27}{16}x^3 + \frac{675}{128}x^4 + \frac{3969}{256}x^5 + \ldots$$

$$E(x) \equiv \frac{H(x) - mc^2}{mc^2}, \qquad x = \left(\frac{GM}{c^3}\Omega\right)^{2/3}$$

$$\text{ISCO:}\quad \frac{dE}{dx} = 0: \quad x_{\text{ISCO}} = \frac{1}{6}$$

$$\text{SBH:}\quad E_{\text{ISCO}} = \sqrt{\frac{8}{9}} - 1 = -\,0.0572, \quad x_{\text{ISCO}}^{3/2} = 0.068$$

$$\text{SBH:}\quad r_{\text{ISCO}} = \frac{6MG}{c^2}$$

$$\text{KBH:}\quad \frac{MG}{c^2} \le r_{\text{ISCO}}^{(+)} \le \frac{6MG}{c^2} \le r_{\text{ISCO}}^{(-)} \le \frac{9MG}{c^2}$$

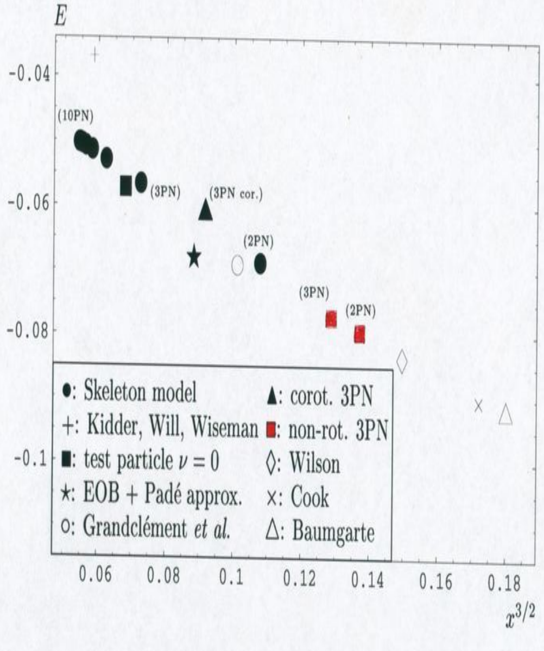
The ISCO for Binary Black Holes

3PN approximation to GR (conservative part):

$$c^2 E_{3PN} \equiv \hat{H}_N + \hat{H}_{[1PN]} + \hat{H}_{[2PN]} + \hat{H}_{[3PN]}$$

$$\begin{aligned} E_{3PN}(x) &= -\frac{x}{2} + \left(\frac{3}{8} + \frac{1}{24}\nu\right) x^2 + \left(\frac{27}{16} - \frac{19}{16}\nu + \frac{1}{48}\nu^2\right) x^3 \\ &+ \left(\frac{675}{128} + \left(-\frac{34445}{1152} + \frac{205}{192}\pi^2\right)\nu + \frac{155}{192}\nu^2 + \frac{35}{10368}\nu^3\right) x^4 \end{aligned}$$

ISCO: $\frac{dE_{3PN}}{dx} = 0$



Remarks

- The difference between 'Wilson' (2PN, vanishing h_{ij}^{TT}) and non-rot. 2PN Einstein is just the contribution from the leading order (2PN) h_{ij}^{TT}. Higher PN order 'Wilson' does not exist.
- An additional extrinsic curvature truncation yields (non-rot.) skeleton 2PN. The skeleton dynamics exists to arbitrary PN orders. But its a kind of model theory only.
- 3PN corot. vs. 3PN non-rot. (from L. Blanchet 2002) shows the influence of rigidly locked proper rotation at 3PN Einstein theory. The difference mainly results from the proper rotational energy of each object ($I\omega^2/2$) and not from the spin-orbit coupling. Corotation, when approaching ISCO, seems not very realistic.
- The Grandglément et al. (numerical, corotation, Misner-Lindquist BHs, vanishing h_{ij}^{TT}, CTS approach) value, in regard to 3PN corot. Einstein, fits with the difference between 2PN Einstein and 2PN Wilson.

- The Cook (numerical, Misner-Lindquist BHs, vanishing h_{ij}^{TT}, CTT approach) and Baumgarte (numerical, Brill-Lindquist BHs, puncture method, vanishing h_{ij}^{TT}, CTT approach) values seem to be too strongly influenced by the 1PN dynamics which is controlled by the Bowen-York extrinsic curvature. In the bodies dynamics, the CTT and CTS approaches are identical through 2PN order inclusively (neither $\partial_t h_{ij}^{TT}$ nor π_{TT}^{ij} contribute).

SKELETON FIELD

Flesh and skeleton

$$
\begin{aligned}
\Psi^{-7}\pi^{i}_{\ j}\pi^{j}_{\ i} &= \Psi^{-7}\pi^{i}_{\ j}\mathrm{STF}(2\partial_i V_j) \\
&= \Psi^{-7}\big(-2V_i\partial_j \underset{\substack{\uparrow \\ \text{skeleton}}}{\pi^{i}_{\ j}} + 2\partial_j(V_i \underset{\substack{\uparrow \\ \text{flesh}}}{\pi^{i}_{\ j}})\big) = \mathcal{O}\left(\frac{1}{c^6}\right)
\end{aligned}
$$

Properties of the flesh term:

- contributes to the 2PN field (and dynamics)
- vanishes when $p_{Ai} = 0$
- vanishes in the test body limit
- contains poles in dimensional regularization
  ← compensated by poles in h^{TT}_{ij} in the Einstein theory

Computation of lapse and shift

Evolution equations for γ_{ij} and π^{ij}

$$\frac{1}{c}\partial_t\gamma_{ij} = \frac{N}{\sqrt{\gamma}}(2\pi_{ij} - \pi^k_{\ k}\gamma_{ij}) + 2D_{(i}N_{j)}$$

$$\begin{aligned}\frac{1}{c}\partial_t\pi^{ij} = -\sqrt{\gamma}\left[N\left(\mathrm{R}^{ij} - \frac{1}{2}\gamma^{ij}\mathrm{R}\right) - D^iD^jN + \gamma^{ij}D_mD^mN\right] \\ +\frac{N}{\sqrt{\gamma}}\left[\pi^{ij}\pi^k_{\ k} - 2\pi^i_{\ k}\pi^{kj} + \frac{1}{2}\gamma^{ij}\left(\pi^{kl}\pi_{kl} - \frac{1}{2}(\pi^k_{\ k})^2\right)\right] \\ -\left[\pi^{kj}D_kN^i + \pi^{ki}D_kN^j - D_k(\pi^{ij}N^k)\right] \\ +\frac{8\pi G}{c^4}N\sum_{A=1}^{N}\frac{p_{Ak}p_{Al}}{m_A}\gamma^{ik}\gamma^{jl}\left(1 + \frac{p_{Am}p_{An}}{m_A^2c^2}\gamma^{mn}\right)^{-\frac{1}{2}}\delta_A\end{aligned}$$

Elliptic equations for N and N^i $\leftarrow$ $\gamma_{ij} = \frac{1}{3}\gamma_{kk}\delta_{ij}$ and $\pi^{ii} = 0$

Skeleton Hamiltonian for 2 black holes

equations satisfied by $\Psi_1 \equiv \Psi_{\mathrm{x}=\mathrm{x}_1}$ and Ψ_2
deduced from the linear independence of the δ_A's

$$\Psi_1 = 1 + \frac{Gm_2}{2r_{12}c^2\Psi_2}\left(1 + \frac{p_2^2}{m_2^2c^2\Psi_2^4}\right)^{\frac{1}{2}} + \frac{Gp_{2i}V_{2i}}{2r_{12}c^3\Psi_2^7}$$

$$\Psi_2 = 1 + \frac{Gm_1}{2r_{12}c^2\Psi_1}\left(1 + \frac{p_1^2}{m_1^2c^2\Psi_1^4}\right)^{\frac{1}{2}} + \frac{Gp_{1i}V_{1i}}{2r_{12}c^3\Psi_1^7}$$

where $\Psi_1 = 1 + \frac{G\alpha_2}{2r_{12}c^2}$ and $r_{12} = |\mathrm{x}_1 - \mathrm{x}_2|$

$$H = -\frac{c^4}{2\pi G}\int d^3\mathrm{x}\,\Delta\Psi = c^2(\alpha_1 + \alpha_2)$$

Skeleton field for 2 black holes I

3-metric

$$\gamma_{ij} = \Psi^4 \delta_{ij} \quad \text{with} \quad \Psi = 1 + \frac{G}{2c^2}\left(\frac{\alpha_1}{r_1} + \frac{\alpha_2}{r_2}\right)$$

Lapse function

flesh-term truncation + projection of the lapse equation on δ_B

$$\boxed{\begin{aligned}\chi_B = 1 - \frac{Gm_A}{r_{12}c^2}\Psi_A^{-4}\chi_A \Bigg\{ & \frac{7p_{Ai}(V_{i\mathrm{x}=\mathrm{x}_A})}{m_A c} \\ & + \left[1 + \frac{p_{Ai}p_{Ai}}{m_A^2 c^2 \Psi_A^4}\right]^{-1/2} \left[3\Psi_A^2 \frac{p_{Ai}p_{Ai}}{m_A^2 c^2} + \Psi_A^6\right]\Bigg\} \\ \text{where} \quad \chi_1 = 1 - \frac{G\beta_2}{2r_{12}c^2} & \end{aligned}}$$

⇒ lapse given by $N = \frac{\chi}{\Psi}$ with $\chi = 1 + \frac{G}{2c^2}\left(\frac{\beta_1}{r_1} + \frac{\beta_2}{r_2}\right)$

Skeleton field for 2 black holes (II)

Shift function

- 2nd skeleton approximation: 2PN flesh term neglected in N^i

$$\partial_j\left(\Psi^{-6}N\pi^i_{\ j}\right) \to \Psi^{-6}N\partial_j\pi^i_{\ j} = -\frac{8\pi G}{c^3}\Psi^{-6}N\sum_{A=1}^{N} p_{Ai}\delta_A$$

- integration of the shift equation ∼ integration of V_i

$$\boxed{N^i = \frac{G}{c^3}\sum_{A=1}^{N}\chi_A\Psi_A^{-7}\left(\frac{1}{2}p_{Aj}\partial_{ij}r_A - 4p_{Ai}\frac{1}{r_A}\right)}$$

Komar mass M_{Komar}:

$$N = 1 - \frac{GM_{\text{Komar}}}{c^2|\mathrm{x}|} + \mathcal{O}\left(\frac{1}{|\mathrm{x}|}\right) \quad \Rightarrow \quad M_{\text{Komar}} = \frac{1}{2}\sum_{A=1}^{2}(\alpha_A + \beta_A)$$

Inspiralling

Quasi-circular inspiralling to order $1/c^{10}$:

$$
\begin{aligned}
- \quad \mu\frac{dE_{\mathrm{circ}}}{dt} = \mathcal{L} = \frac{32c^5}{5G}\nu^2 x^5 [1 + (-\frac{1247}{336} - \frac{35}{12}\nu)x + 4\pi x^{3/2} \\
+ \quad (-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2)x^2 \; - \; (\frac{8191}{672} + \frac{535}{24}\nu)\pi x^{5/2}]
\end{aligned}
$$

$$
\begin{aligned}
x \; &= \; \frac{1}{4}\tau^{-1/4}[1 + (\frac{743}{4032} + \frac{11}{48}\nu)\tau^{-1/4} - \frac{1}{5}\pi\tau^{-3/8} \\
+ \; &\; (\frac{19583}{254016} + \frac{24401}{193536}\nu + \frac{31}{288}\nu^2)\tau^{-1/2} - (\frac{11891}{53760} - \frac{29}{1920}\nu)\pi\tau^{-5/8}]
\end{aligned}
$$

$$\tau = \frac{\nu c^3}{5Gm}(t_c - t)$$

BINARY BLACK–HOLE DYNAMICS AT THE 3.5PN ORDER IN THE ADM FORMALISM

Dissipative Hamiltonian $H^{\mathrm{int}}_{2.5\mathrm{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a, t)$

$$H^{\mathrm{int}}_{2.5\mathrm{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a, t) = 5\pi\, \dot{\chi}_{(4)ij}(t)\, \chi_{(4)ij}(\boldsymbol{x}_a, \boldsymbol{p}_a)$$

with

$$\begin{aligned}\chi_{(4)ij}(\boldsymbol{x}_a, \boldsymbol{p}_a) := \frac{1}{60\pi}\Bigg[& \sum_a \frac{2}{m_a}(\boldsymbol{p}_a^2 \delta_{ij} - 3p_{ai}p_{aj}) \\ & + \frac{1}{16\pi}\sum_a \sum_{b \neq a} \frac{m_a m_b}{r_{ab}}(3n^i_{ab}n^j_{ab} - \delta_{ij})\Bigg]\end{aligned}$$

Dissipative Hamiltonian $H^{\mathrm{int}}_{3.5\mathrm{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a, t)$

$$\begin{aligned}H^{\mathrm{int}}_{3.5\mathrm{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a, t) = {} & 5\pi\, \chi_{(4)ij}(\boldsymbol{x}_a, \boldsymbol{p}_a)\left[\dot{\Pi}_{1ij}(t) + \dot{\Pi}_{2ij}(t) + \ddot{\Pi}_{3ij}(t)\right] \\ & + 5\pi\, \dot{\chi}_{(4)ij}(t)\left[\Pi_{1ij}(\boldsymbol{x}_a, \boldsymbol{p}_a) + \widetilde{\Pi}_{2ij}(\boldsymbol{x}_a, t)\right] \\ & - 5\pi\, \ddot{\chi}_{(4)ij}(t)\Pi_{3ij}(\boldsymbol{x}_a, \boldsymbol{p}_a) \\ & + \dot{\chi}_{(4)ij}(t)\left[{Q'}_{ij}(\boldsymbol{x}_a, \boldsymbol{p}_a, t) + {Q''}_{ij}(\boldsymbol{x}_a, t)\right] \\ & + \frac{\partial^3}{\partial t^3}\left[R'(\boldsymbol{x}_a, \boldsymbol{p}_a, t) + R''(\boldsymbol{x}_a, t)\right]\end{aligned}$$

$\Pi_{2ij}(\boldsymbol{x}_a, \boldsymbol{p}_a)$ in $H^{\text{int}}_{3.5\text{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a, t)$

$$\begin{aligned}
\Pi_{2ij}\left(\boldsymbol{x}_a, \boldsymbol{p}_a\right) := {} & \frac{1}{5}\left(\frac{1}{16\pi}\right)^2 \sum_a \sum_{b\neq a} \frac{m_b}{m_a r_{ab}} \left\{ \left[5(\boldsymbol{n}_{ab}\cdot\boldsymbol{p}_a)^2 - \boldsymbol{p}_a^2\right]\delta_{ij} - 2p_{ai}p_{aj} \right. \\
& \left. + \left[5\boldsymbol{p}_a^2 - 3(\boldsymbol{n}_{ab}\cdot\boldsymbol{p}_a)^2\right] n^i_{ab} n^j_{ab} - 6(\boldsymbol{n}_{ab}\cdot\boldsymbol{p}_a)(n^i_{ab}p_{aj} + n^j_{ab}p_{ai}) \right\} \\
& + \frac{6}{5}\left(\frac{1}{16\pi}\right)^3 \sum_a \sum_{b\neq a} \frac{m_a^2 m_b}{r_{ab}^2}\left(3n^i_{ab}n^j_{ab} - \delta_{ij}\right) \\
& + \frac{1}{10}\left(\frac{1}{16\pi}\right)^3 \sum_a \sum_{b\neq a} \sum_{c\neq a,b} m_a m_b m_c \left\{ \left[5\frac{r_{ca}}{r_{ab}^3}\left(1 - \frac{r_{ca}}{r_{bc}}\right) + \frac{13}{r_{ab}r_{ca}} - \frac{40}{r_{ab}s_{abc}}\right]\delta_{ij} \right. \\
& + \left[3\frac{r_{ab}}{r_{ca}^3} + \frac{r_{bc}^2}{r_{ab}r_{ca}^3} - \frac{5}{r_{ab}r_{ca}} + \frac{40}{s_{abc}}\left(\frac{1}{r_{ab}} + \frac{1}{s_{abc}}\right)\right] n^i_{ab}n^j_{ab} \\
& \left. + \left[2\frac{(r_{ab} + r_{ca})}{r_{bc}^3} - 16\left(\frac{1}{r_{ab}^2} + \frac{1}{r_{ca}^2}\right) + \frac{88}{s_{abc}^2}\right] n^i_{ab}n^j_{ca} \right\},
\end{aligned}$$

with $s_{abc} := r_{ab} + r_{bc} + r_{ca}$,

Rel. acceleration associated with $H^{\text{int}}_{2.5\text{PN}}$ and $H^{\text{int}}_{3.5\text{PN}}$

$$\begin{aligned}
\boldsymbol{a} = {} & -\frac{GM}{r^2}\boldsymbol{n} + \frac{1}{c^2}\left\{ \frac{GM}{r^2}\left\{ \left[-(1+3\nu)\boldsymbol{v}^2 + \frac{3}{2}\dot{r}^2\nu\right]\boldsymbol{n} + (4\dot{r} - 2\dot{r}\nu)\boldsymbol{v} \right\} + \frac{G^2M^2}{r^3}(4+2\nu)\boldsymbol{n} \right\} \\
& + \frac{1}{c^5}\left\{ \frac{G^2M^2}{r^3}\left[\left(-24\dot{r}^3\nu + \frac{96}{5}\dot{r}\boldsymbol{v}^2\nu\right)\boldsymbol{n} + \left(\frac{64}{5}\dot{r}^2\nu - \frac{88}{15}\boldsymbol{v}^2\nu\right)\boldsymbol{v}\right] + \frac{G^3M^3}{r^4}\left(\frac{16}{5}\dot{r}\nu\boldsymbol{n} - \frac{8}{15}\nu\boldsymbol{v}\right) \right\} \\
& + \frac{1}{c^7}\left\{ \frac{G^2M^2}{r^3}\left\{ \left[-46\dot{r}^5\nu + 24\dot{r}^5\nu^2 + \boldsymbol{v}^4\left(-\frac{138}{35}\dot{r}\nu - \frac{516}{35}\dot{r}\nu^2\right) + \boldsymbol{v}^2\left(56\dot{r}^3\nu - \frac{4}{7}\dot{r}^3\nu^2\right)\right]\boldsymbol{n} \right. \right. \\
& + \left. \left[\frac{334}{7}\dot{r}^4\nu - \frac{268}{7}\dot{r}^4\nu^2 + \boldsymbol{v}^4\left(\frac{1006}{105}\nu - \frac{64}{105}\nu^2\right) + \boldsymbol{v}^2\left(-\frac{2356}{35}\dot{r}^2\nu + \frac{148}{5}\dot{r}^2\nu^2\right)\right]\boldsymbol{v} \right\} \\
& + \frac{G^3M^3}{r^4}\left\{ \left[\frac{10188}{35}\dot{r}^3\nu + \frac{324}{7}\dot{r}^3\nu^2 + \boldsymbol{v}^2\left(-\frac{18656}{105}\dot{r}\nu - \frac{1116}{35}\dot{r}\nu^2\right)\right]\boldsymbol{n} + \left[-\frac{17308}{105}\dot{r}^2\nu - \frac{244}{21}\dot{r}^2\nu^2 \right. \right. \\
& \left. \left. \left. + \boldsymbol{v}^2\left(\frac{4394}{105}\nu - \frac{16}{35}\nu^2\right)\right]\boldsymbol{v} \right\} + \frac{G^4M^4}{r^5}\left[\left(-\frac{152}{15}\dot{r}\nu - \frac{632}{105}\dot{r}\nu^2\right)\boldsymbol{n} - \left(\frac{386}{105}\nu + \frac{16}{15}\nu^2\right)\boldsymbol{v}\right] \right\}
\end{aligned}$$

Outline

- Binary black–hole dynamics at the 3.5PN order in the ADM formalism, Phys. Rev. D **68**, 044004 (2003)
- Post–Newtonian accurate parametric solution to the dynamics of spinning compact binaries in eccentric orbits: I. The leading order spin–orbit interaction, submitted to Phys. Rev. D