Abstract
Massive binary systems (neutron stars or black holes) in the nearby universe are one of the more likely events to be observed by the first generation of gravitational wave detectors. This talk outlines recent progress in the numerical simulation of black hole spacetimes, in particular focusing on evolutions of binaries in close quasi-circular orbit, and the extraction of physical information from horizon dynamics and far-zone wave indicators.
Overview

- Motivation: gravitational waves from binary black holes
- Singularity excision
- Horizon finding
- Gravitational wave extraction
- Black hole simulations: single black holes, grazing collisions, plunges from orbital configurations
Binary systems

- Strong, dynamical gravitational fields $\rightarrow$ gravitational waves

- A physical system of particular interest is the inspiral and merger of dense compact binaries: neutron stars (NS) and black holes (BH)

- Science from observed inspirals:
  - Obtainable information: masses, spins, distance, location
  - Frame dragging $\rightarrow$ precession $\rightarrow$ modulation
  - Wave tails, limits on alternative field theories (Brans-Dicke), graviton mass
  - Correspondence with EM counterpart: $\gamma$-ray burst?, relative speed of speed of light and gravitational waves
  - NS binaries: wave cutoff $\rightarrow$ radius $\rightarrow$ eq. of state
  - standard candles – Hubble constant

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Event rates for binary BHs

- NS/NS binaries most reliably understood source in terms of event rates (there are actual observations)

- NS/BH and BH/BH event rates are estimated from population synthesis models [Kalogera et al.]

<table>
<thead>
<tr>
<th>Model</th>
<th>1st Generation Detectors</th>
<th>2nd Generation Detectors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Range</td>
<td>Event rate</td>
</tr>
<tr>
<td>NS/NS ($1.4M_\odot/1.4M_\odot$)</td>
<td>20 Mpc</td>
<td>1/3000 yrs – 1/3 yrs</td>
</tr>
<tr>
<td>NS/BH ($1.4M_\odot/10M_\odot$)</td>
<td>43 Mpc</td>
<td>≥ 1/2500 yrs to 1/2 yrs</td>
</tr>
<tr>
<td>BH/BH ($10M_\odot/10M_\odot$)</td>
<td>100 Mpc</td>
<td>1/300 yrs – 1/yr</td>
</tr>
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[Cutler, Thorne 2002]  
[Flanagan, Hughes 1998]
• Black holes captured gravitationally – potentially highly elliptical orbits

• Gravitational radiation reaction circularises the orbits

• Black holes gradually inspiral, losing energy

• Eventually, nonlinear effects dominate and black holes plunge together

• A single perturbed black hole is formed, and rings down
Methods for numerical relativity: Summary

- Numerical methods hold the promise to provide solutions which are arbitrarily accurate (depending on computational resources) and completely general, corresponding to any given initial data source.

- Stability of numerical codes has been the main limiting factor.

- Formulation of the Einstein equations crucial: In recent years, strongly hyperbolic formulations (eg. BSSN, KST, Z4) have lead to significant progress.

- Gauge needs to be determined with care:
  - Singularity avoiding slicings
  - Minimal coordinate distortion
  - Remove dynamics (eg. co-rotation)

- A number of specialised techniques, tools, and numerical methods are needed for black hole space times:
  - Singularity excision
  - Horizon finding
  - Wave extraction
**Excision**

- The region of the grid inside the event horizon is causally disconnected from the region outside.

- Thus we can try to cut a region from the grid inside the horizon – excise the singularity.

- The causality of the spacetime indicates that any timelike boundary within the horizon is an inflow boundary.

- In practice, at least stability can be influenced by the excision bc.

- Numerical domain of dependence not the same as the physical domain of dependence – errors can leak out.

- Requires consistent finite differencing at the excision boundary – “body fitting” coordinates are preferable to “lego” spheres.
Horizon finding

The “location” of a black hole is defined by the presence of a horizon. It is a non-trivial computation to determine horizon shapes in dynamical spacetimes.

**Event horizons (EH):**
- “true” location of the black hole
- only defined globally – can only be computed once the entire spacetime evolution is known
- determined as a post-processing step, by evolving null surfaces backwards in time on the geometry determined by the simulation

**Apparent horizons (AH):**
- locally defined in terms of ingoing light cones – can compute during a simulation
- regarded as a good approximation to the BH location – always interior to the EH
- recently developed “dynamical horizon” formalism allows determination of various physical quantities, spins, infalling energy.
Wave extraction

Zerilli extraction
In the weak field region, say \( r > \lambda \), we assume the numerically generated spacetime is a perturbation of a background Schwarzschild metric. Integrate multipole amplitudes over a sphere at a fixed extraction radius.

Newman-Penrose form of Weyl tensor components
Radiation is encoded in the Weyl component

\[
\Psi_4 = -C_{\alpha\beta\gamma\delta} \bar{m}^\alpha n^\beta \bar{m}^\gamma n^\delta
\]
defined relative to a null tetrad. This is a frame dependent measure. We generally assume the spacetime is close to Kerr in the wave zone and use the Kinnersley tetrad.

[Alcubierre et al. 2001]
Grazing collisions

Brandt et al. 2000:

- Spinning black holes, mass $M$, boosted together from $R = (\pm 5M, \pm M, 0)$, impact parameter $2M$
- Initial data constructed from “superposed” Kerr-Schild black holes
- Considered 3 cases: (1) $a = 0.5M$ aligned, (2) $a = 0$, (3) $a = 0.5M$ anti-aligned
- Use ADM formalism, Kerr-Schild lapse and shift
- Find common apparent horizon at $t \approx 4M$, simulations last to $t \approx 30M$

Alcubierre et al. 2001:

- Closely separated, unequal mass, puncture black holes with misaligned spins
  
  $R_1 = (0, 1.5M, 0)$  
  $R_2 = (0, -1.5M, 0)$  
  $P_1 = (2M, 0, 0)$  
  $R_2 = (-2M, 0, 0)$  
  $S_1 = (-M^2/2, 0, -M^2/2)$  
  $S_2 = (0, M^2, -M^2)$

- BSSN, “$1 + \log$” slicing, no shift, no excision

(In neither case is it clear that the initial data corresponds to separate black holes)
Initial value problem for binary sources

Initial data corresponds to a solution to the Einstein constraint equations on the initial $t = \text{constant}$ slice.

Computational cost, plus lack of numerical stability of simulations mean that we would like to begin simulations as deep into the potential well as possible.

ie. we aim to simulate the last few orbits before merger

Unfortunately, solutions for closely separated binary black holes in quasi-circular orbit are not known.

A number of methods for generating approximate solutions have been proposed.
Initial value constraints

Constraint equations need to be solved on the initial slice. For the commonly used 3 + 1 formalism, these amount to:

\[ \mathcal{H} = R + K^2 - K_{ij}K^{ij} = 0 \]  \hspace{0.5cm} \text{(hamiltonian)}

\[ \mathcal{M}_a = \nabla^i (K_{ai} - \gamma_{ai}K) = 0 \]  \hspace{0.5cm} \text{(momentum)}

(1) York-Lichnerowicz decomposition

Specify the conformal metric \( \tilde{\gamma}_{ij} \) and a conformal transverse traceless tensor \( M_{ij} \) as free data.

Momentum constraint decouples from the Hamiltonian constraint. Analytic solution is given by the Bowen-York extrinsic curvature in terms of boost \( P^a \) and spin \( S^a \):

\[ K^{ab} = \frac{3}{r^2} (P^{(a}n^{b)} - (g^{ab} - n^a n^b)P^c n_c) + \frac{3}{r^3} (e^{acd} S_c n_d n^b + e^{bcd} S_c n_d n^a). \]

Under assumption of conformal flatness and maximal slicing the Hamiltonian constraint takes the form:

\[ \tilde{\nabla}^2 \psi - \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0, \]

which should be solved for \( \psi \).
Initial value constraints

Constraint equations need to be solved on the initial slice. For the commonly used 3 + 1 formalism, these amount to:

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\[ \mathcal{M}_a = \nabla^i (K_{ai} - \gamma_{ai} K) = 0 \]  

(2) Conformal thin sandwich

Free data are the conformal metric and it’s “velocity”:

\[ \tilde{\gamma}_{ij} := \psi^{-4} \gamma_{ij}, \quad \tilde{u}_{ij} := \partial_t \tilde{\gamma}_{ij} \]

Constraint equations determine \( \psi \) and the shift \( \beta \).

Stationary of the initial data can be cleanly specified by setting the free data \( \tilde{u}_{ij} = 0 \).
Data for quasi-circular orbits

- The binary systems of most interest are in late stages of quasi-circular inspiral.

- For a given separation, we require an initial momentum that will place the bodies in such orbits.

(1) Effective potential method

- Cook (1994) carried out a search for potential “quasi-circular” orbit parameters as minima of a binding energy:

\[ E_{\text{eff}} = M_{\text{ADM}} - \sum M_{\text{AH}} \]

- Effect potential defined by the difference between asymptotically defined ADM mass, and local horizon masses

- Considered non-spinning, equal mass binaries determined via conformal imaging formalism

- Baumgarte (1999) applied the method to Brill-Lindquist punctures, found a similar ISCO
Data for quasi-circular orbits

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(2) Helical Killing Vector approximation

In addition to solving the constraints on the initial surface, enforce an additional assumption that there exists a helical killing vector [Gourgoulhon, Grandclement, Bonazzola 2002]

\[ \ell = \frac{\partial}{\partial t_0} + \Omega \frac{\partial}{\partial \phi_0}. \]

The stationarity of the solution is naturally expressed in terms of the free data for the thin-sandwich formalism: \( \tilde{u}_{ij} = 0 \)

Recently, a similar HKV ansatz has been applied to puncture data, yielding essentially identical orbital parameters as the effective potential estimate [Tichy, Brügmann 2003].
Data for quasi-circular orbits

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![Graph showing binding energy as a function of orbital velocity](image)

[Damour, Gourgoulhon, Grandclement 2002]
Data for quasi-circular orbits

- Current approximations for BBH initial data rely on the assumption that a circular orbit is a reasonable approximation, even for closely separated binaries.

- This assumption was analysed by Miller (2003):
  - Determine PN waveform from inspiral from 1000M
  - Compare with PN inspiral from 10M, using circular orbit approximation

- For the circular approximation is only effective for binaries more than 6 orbits before plunge
Black hole simulations

Single black hole [Alcubierre & Brügmann 2000]:

- Consider Schwarzschild in Eddington-Finkelstein coordinates (ie. manifestly static)
- Singularity avoiding “1 + log” slicing, \( \Gamma \)-driver shift
- Imposed octant symmetry – required for stability
  - \( Y \), Baumgarte, Shapiro 2002 demonstrated removed this condition
- Cubical excision region
- Were able to evolve “forever” (\( > 100000M \))
- Followed by evolutions of distorted BH+Brill wave [Alcubierre et al. 2001]

Single black hole [Scheel et al. (2002)]:

- Considered Schwarzschild black hole in Painlevé-Gullstrand coordinates
- Kidder-Scheel-Teukolsky (KST) formalism
- Densitised lapse, fixed shift from exact solution
- Pseudo-spectral representation of functions on spatial slices
- Evolution times \( t > 8000M \) for appropriately tuned KST parameters

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Black hole simulations

Head-on collision [Alcubierre et al. (2002)] :
- Pair of equal mass, non-spinning Brill-Lindquist “puncture” black holes at \((0, \pm 1.15M, 0)\)
- BSSN equations, \(1 + \log\) lapse
- Dynamic \(\Gamma\)-driver shift condition, reduces grid stretching
- Stable simulation lasts to \(> 5000M\)

Moving black hole [Sperhake et al. (2003)] :
- Single Eddington-Finkelstein black hole, moved via coordinate transformation
- BSSN evolution system, \(1 + \log\) lapse, analytic shift
- Excision boundary implemented via extrapolations, points repopulated behind the moving BH
Convergence testing

An increase in resolution should result in an increase in accuracy which is determined by the finite difference order:

$$\frac{\partial \phi}{\partial x} \bigg|_i = \frac{\phi|_{i+1} - \phi|_{i-1}}{2\Delta x} + O(\Delta x^2) \quad (2\text{nd order})$$

For a 2nd order code, doubling the resolution should result in a numerical solution which is four times closer to an exact solution.

If no exact solution is available, another doubling of resolution determines the convergence factor:

$$2^n = \frac{\psi_{hi} - \psi_{med}}{\psi_{med} - \psi_{low}}$$

← Extracted Zerilli waveforms from a head-on collision of two black holes from initial separation $l = 4.7M$. 

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Recent Meudon and puncture evolutions

- **Puncture runs**
  - equal mass, non-rotating puncture BHs
  - conformally flat, bowen-york extrinsic curvature
  - orbital parameters based effective potential sequence
    [Cook (1994) Baumgarte (2000)]

- **Meudon thin sandwich runs**
  - equal mass, co-rotating BHs
  - conformally flat, thin sandwich
  - orbital parameters determined by helical killing vector (HKV)
    approach [Grandclement, Gourgoulhon, Bonazolla (2002)]
Thin sandwich HKV evolutions

- Initial data generated using Lorene spectral solver, provided by Meudon group within EU Network collaboration
- BH interiors filled using isometry condition, then excised [Koppitz, PhD 2004]
- Co-rotating coordinates implemented via a shift vector
- Runs with fixed mesh refinement, typically 5 levels, coarsest $dx = 0.35M$, finest $dx = 0.02M$
- Data shown for initial separation $L = 6.9M$ (outside ISCO)

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Long-lived puncture evolutions

- Punctures evolved from QC-orbit parameters, coordinate separation $d = 6.0M$ (effective potential ISCO is $d = 2.2$, PN ISCO $d = 3.8$).

- Used BSSN evolution system, 1+log and Gamma-freezing shift, fixed mesh refinement.

- Evolution continued past $t = 114M$, the expected orbital period, without finding a common apparent horizon.
Summary

- Binary mergers are among the more likely candidates for early detection; accurate merger simulations can help the detection process and interpretation.

- Stability is the major issue hampering progress in vacuum simulations.

- Current best simulations have been able to evolve stably for the timescale of their expected orbital period.

- The stability of these runs has improved dramatically with careful consideration of the equation formulation and numerical methods.

- Useful physics (e.g., gravitational wave extraction) will require additional improvements: long-term stability, better outer boundary conditions.